SCECGS REDLANDS

TRIAL HIGHER SCHOOL CERTIFICATE 1995

FORM 12

MATHEMATICS

3/4 UNIT - COMMON PAPER

3 UNIT - ADDITIONAL PAPER

TIME ALLOWED: Two hours plus five minutes reading time.

DIRECTIONS TO CANDIDATES:

1. ALL questions may be attempted.
2. ALL questions are of equal value (marks are indicated).
3. Answer each question in a separate Writing Booklet. Write your Candidate number AND Question number on the corner of each Writing booklet.
4. Show all necessary working. Marks may be deducted for careless or badly arranged work.
5. Standard integrals are printed on the last page of this paper.
6. Board-Approved Calculators may be used.
7. You may ask for extra Writing Booklets, if you need them.
8. Students are advised that this is a TRIAL EXAMINATION only and cannot in any way guarantee the content or the format of the HIGHER SCHOOL CERTIFICATE EXAMINATION. We hope that this paper will provide a positive contribution to your preparation for the final Examinations.

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1995

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MARKING SCHEDULE

QUESTIONS	MARKS FOR SECTIONS					TOTAL
	a	b	c	d	e	
Question No. 1	2	4	2	3	4	15
Question No. 2	4	7	4	-	-	15
Question No. 3	5	5	5	-	-	15
Question No. 4	5	10	-	-	-	15
Question No. 5	11	4	-	-	-	15
Question No. 6	6	9	-	-	-	15
Question No. 7	9	6	-	-	-	15

QUESTION 1
(15 marks)

Use a separate Writing Booklet.

(a) Solve $2x^2 - x - 6 < 0$.

(b) Show that $\dfrac{d}{dx}\left(\dfrac{2x}{\sqrt{(2x^2 + 1)}}\right) = \dfrac{2}{(2x^2 + 1)^{3/2}}$

(c) Find the exact value of

$$\int_0^{\sqrt{3}} \frac{dx}{\sqrt{(3 - x^2)}}$$

(d) Evaluate

$$\int \frac{\text{Cos}(\log_e x)}{x}\,dx$$

by using the substitution $t = \log_e x$.

(e) For what values of x is $\dfrac{1}{x - 2} \leq \dfrac{1}{2}$

Indicate your solution on a number line.

QUESTION 2
(15 marks)

Use a separate Writing Booklet.

(a) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2 2x \, dx$

(b) The acceleration of a particle moving in a straight line is given by:

$$\ddot{x} = 3 - 4x,$$

where x is the displacement in metres, from the origin and t is the time in seconds.

If the particle starts from rest at $x = 1$ metres,

(i) Show that the velocity of the particle is given by

$$v^2 = 2(-2x^2 + 3x - 1).$$

(ii) Identify the second position where the particle will come to rest.

(iii) What will be the acceleration at the second position where the particle comes to rest.

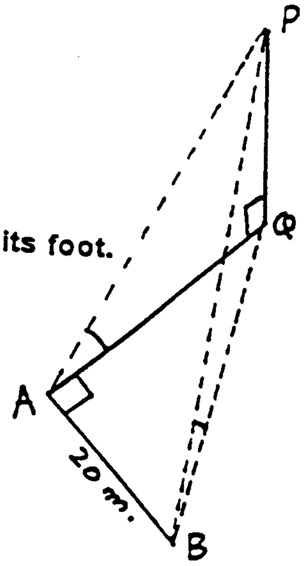
(c) Use Newton's method to find a second approximation to a root of

$$x^3 - 3x - 10 = 0.$$

Take $x = 2.7$ as the first approximation.
Give your answer to two decimal places.

QUESTION 3
(15 marks)

Use a separate Writing Booklet.



(a) PQ is a tower standing on a horizontal plane. Q, being its foot.

A and B are two points on the plane such that
$\angle QAB = 90°$ and $AB = 20$ metres

It is found that $\cot P\hat{A}Q = \frac{3}{10}$ and $\cot P\hat{B}Q = \frac{1}{2}$

(i) Copy the diagram to illustrate this information into your Writing Booklet.

(ii) Find the height of this tower.

(b) Consider the function $f(x) = \frac{1}{2}\cos^{-1}(\sqrt{3}\,x)$

(i) Evaluate $f\left(\frac{1}{2}\right)$

(ii) State the domain of $f(x)$.

(iii) For what values of $f(x)$ is this function defined?

(iv) Draw a graph of $f(x)$, labelling any key features.

(c) (i) Lenny Wu establishes a fund for his daughter with a deposit of \$200, at 9% p.a. interest compounded monthly. To how much money would this amount accrue at the end of 3 years?

(ii) Suppose, at the beginning of each subsequent month after the first deposit has been made, a further \$200 had been added to the fund and had also earned 9% p.a. interest,. compounded monthly, how much money would there be in the fund after 12 years?

QUESTION 4 Use a separate Writing Booklet.
(15 marks)

(a) Prove by mathematical induction that

$$2 + 2^2 + 2^3 + \ldots\ldots + 2^n = 2(2^n - 1)$$

(b) A watertank, in the form of an inverted cone with semi-vertical angle 30°, contains water to a depth of h metres as shown in the diagram.

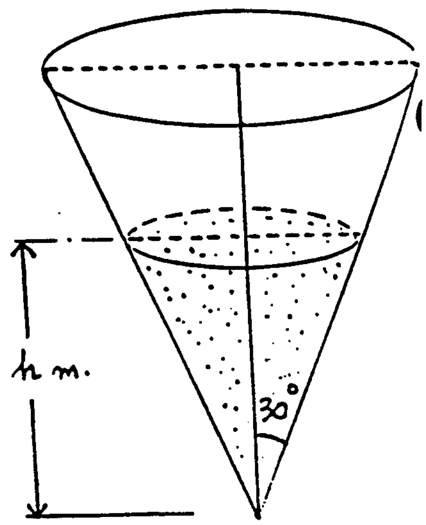
(i) Copy the diagram into your answer booklet.

(ii) Show that the volume of water in the tank is given by
$$V = \frac{\pi}{9} h^3 \text{ metres}^3.$$

(iii) The vertex of the tank has now sprung a leak and water in leaking out at the rate of $0.2\sqrt{h}$ metre^3 / minute.

Find an expression in terms of h, for the rate at which the depth of water is being reduced.

(iv) What will be the rate of reduction in depth when $h = 4$ metres? (Leave your answer in terms of π).



QUESTION 5 Use Separate Writing Booklet
(15 marks)

(a) Two points $P(4p, 2p^2)$ and $Q(4q, 2q^2)$ lie on the parabola $x^2 = 8y$.

(i) Show that the equation to the tangent to the parabola at P is $y = px - 2p^2$.

(ii) The tangent at P and the line through Q, parallel to the Y-axis, intersect at T. Find the Co-ordinates of T.

(iii) Find the Co-ordinates of M, the mid-point of PT.

(iv) Determine the Cartesian Equation of the locus of M when $pq = -1$.

(b) Write the most general solution for θ, when $2\sin^2\theta + 3\cos\theta = 0$.

QUESTION 6. (15 marks) Use a separate Writing Booklet

(a) The population of sheep on a farm is given by the relation

$$\frac{dN}{dt} = k(N - 2000),$$

when N is the number of sheep at any time t and k is a constant. If there are initially 5000 sheep and after 2 years, there are 6000 sheep,

(i) Show that $N = 2000 + 3000e^{kt}$ for some value of k.

(ii) Give a value for k correct to eight decimal places.

(iii) Show also that the number of sheep on the farm after 3 years is 6618.

(iv) After how many years will the sheep population reach 10,000? (Give your answer to the nearest whole year).

(b) A particle moves in a straight line so that its position $x$cm from a fixed point 0, at time t seconds is given by:

$$x = 4\sin 2t + 3\cos 2t$$

(i) Show that this motion can be expressed in the form $x = A\sin(2t + \alpha)$ where $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$

Evaluate A and α.

(ii) Prove that the motion is Simple Harmonic.

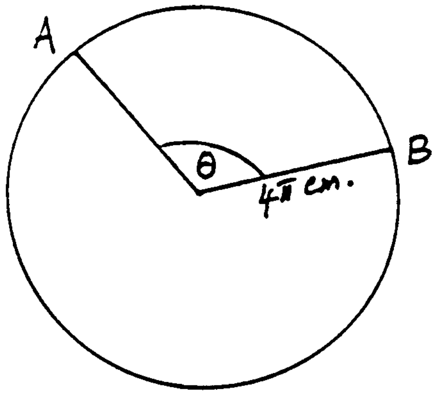
(iii) What is the Period of Oscillation?

(iv) Determine its maximum displacement.

(v) Calculate its velocity when x is 3cm from 0.

QUESTION 7 (15 marks) Use a separate Writing Booklet

(a)



A sector AB of angle θ radians is cut from a circular disc of radius 4π cm and used to make the complete curved surface of a right circular cone (with no overlap).

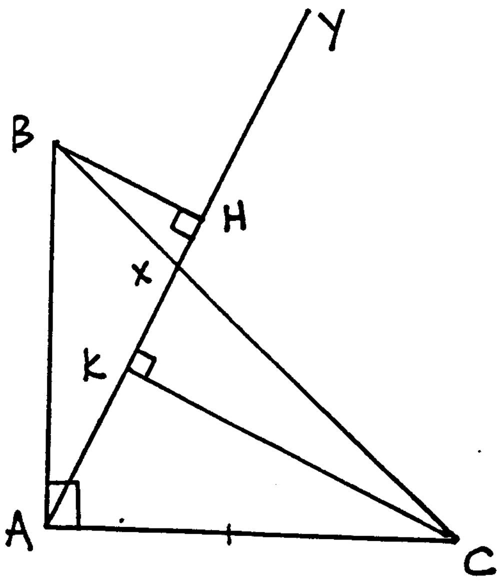
(i) Show that the volume of this cone is given by

$$V = \frac{8\pi\theta^2}{3}\sqrt{4\pi^2 - \theta^2}$$

(ii) Find the value of θ for which the volume of this cone is a maximum. (Make sure that the maximum is verified).

(b)

NOT TO SCALE



In $\triangle ABC$, AB = AC and $\angle BAC = 90°$. AY is any straight line through A. BH and CK are perpendiculars from B and C to AY.

(i) Copy the given diagram into your answer booklet.

(ii) Prove that AH = CK.

END OF PAPER

SOLUTIONS

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EXAMINERS :
Mr. John — Question Nos. 1 & 6(a)
Mr. Habkouk — Question Nos. 2 & 6(b)
Mr. Dharma — Question Nos. 3 & 7(a)
Mrs. Taylor — Question Nos. 4 & 7(b)

FORM 12, 3/4 UNIT PAPER
H.S.C TRIALS
JULY 1995

SOLUTIONS

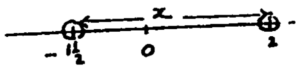
The following are only one set of solutions
{Alternative solutions are acceptable.
{Alternative solutions will be marked on its own merits.
The maximum score for each question is 15 marks

Question No. 1

(a) Solve $2x^2 - x - 6 < 0$

ie $(2x+3)(x-2) < 0$

$\therefore -1\tfrac{1}{2} < x < 2$



(b). Let $y = \dfrac{2x}{\sqrt{2x^2+1}}$

$= \dfrac{2x}{(2x^2+1)^{1/2}}$

$\therefore \dfrac{dy}{dx} = \dfrac{(2x^2+1)^{\frac{1}{2}}\cdot 2 - 2x\cdot\frac{1}{2}(2x^2+1)^{-\frac{1}{2}}\cdot 4x}{2x^2+1}$

$= \dfrac{2\sqrt{2x^2+1} - \dfrac{4x^2}{\sqrt{2x^2+1}}}{(2x^2+1)}$

$= \dfrac{2(2x^2+1) - 4x^2}{(2x^2+1)^{3/2}}$

$= \dfrac{4x^2+2-4x^2}{(2x^2+1)^{3/2}}$

$= \dfrac{2}{(2x^2+1)^{3/2}}$ //

Alternatively:
you may re-write the question as:
$y = 2x\,(2x^2+1)^{-\frac{1}{2}}$
and then use the Product Formulas

(c) $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{\sqrt{3-x^2}}$

$= \left[\sin^{-1}\dfrac{x}{\sqrt{3}}\right]_0^{\sqrt{3}}$

$= \left(\sin^{-1}\dfrac{\sqrt{3}}{\sqrt{3}}\right) - \sin^{-1} 0$

$= \dfrac{\pi}{2}$ //

(d) $I = \displaystyle\int \frac{\cos(\log_e x)}{x}\,dx$

Let $t = \log_e x$

$\therefore dt = \dfrac{1}{x}\,dx$

$\therefore dx = x\,dt$

$\therefore I = \displaystyle\int \frac{\cos t}{\cancel{x}}\cdot \cancel{x}\,dt$

$= \displaystyle\int \cos t\,dt$

$= \sin t + c$

$= \sin(\log_e x) + c$

Remember: $x > 0$

or $I = \sin \ln|x| + c$

(e). Solve $\dfrac{1}{x-2} \leq \dfrac{1}{2}$

This is not valid for $x = 2$
(because $x = 2$ gives an infinite value)

Method I

Let $\dfrac{1}{x-2} = \dfrac{1}{2}$

$\therefore x - 2 = 2$

$\therefore x = 4$

So, use $x = 2$ and $x = 4$ as boundaries:

When $x < 2$, (say $x = 1$), $\frac{1}{-1} < \frac{1}{2}$ True
$x = 2$ is not possible
For $2 < x < 4$, (say $x = 3$), $\frac{1}{1} \not< \frac{1}{2}$
When $x = 4$, $\frac{1}{2} = \frac{1}{2}$ — True
When $x > 4$, (say $x = 5$), $\frac{1}{3} < \frac{1}{2}$ True

$\therefore$ The solution is:

ie $x < 2$
and $x \geq 4$

NOTE: Another Method is the SQUARING Method (or Method of Squaring)

$\dfrac{1}{x-2}\times(x-2)^2 \leq \dfrac{1}{2}(x-2)^2$

$\therefore 2(x-2) \leq x^2 - 4x + 4$

$\therefore 0 \leq x^2 - 6x + 8$

ie: $x^2 - 6x + 8 \geq 0$

ie $(x-4)(x-2) \geq 0$: But $x \neq 2$

$\therefore$ The solution is:
$x \geq 4$ and $x < 2$

Yet Another Method is The Method of Graphing:

Question no. 2

(a) Evaluate $I = \int_0^{\pi/2} \cos^2 2x \, dx$.

$\cos 4x = 2\cos^2 2x - 1$

$\therefore \cos^2 2x = \frac{1}{2}(1 + \cos 4x)$

$\therefore I = \frac{1}{2}\int_0^{\pi/2} (1 + \cos 4x)\,dx$

$= \frac{1}{2}\left[x + \frac{1}{4}\sin 4x\right]_0^{\pi/2}$

$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{1}{4}\sin 2\pi\right) - (0)\right]$

$= \frac{\pi}{4}$ //

(b). $acc^n = \ddot{x} = 3 - 4x$

(i) ie $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 3 - 4x$

$\therefore \int \frac{d}{dx}\left(\frac{1}{2}v^2\right)dx = \int (3-4x)\,dx + c$

$\therefore \frac{1}{2}v^2 = 3x - 2x^2 + c$ ——— (1)

The particle starts from rest at x = 1 metre.

ie $v = 0$ when $x = 1$

$\therefore$ sub in (1), $0 = 3 - 2 + c$

$\therefore c = -1$

$\therefore \frac{1}{2}v^2 = 3x - 2x^2 - 1$

ie $v^2 = 2(3x - 2x^2 - 1)$

ie $v^2 = 2(-2x^2 + 3x - 1)$

(ii) When the particle comes to rest $v = 0$

$\therefore 2x^2 - 3x + 1 = 0$

ie $(2x-1)(x-1) = 0$

$\therefore x = 1, \frac{1}{2}$

$\therefore$ The particle will come to rest again when $x = \frac{1}{2}$ m.

(iii) So, the acceleration when the particle comes to rest at $x = \frac{1}{2}$ can be obtained by substituting $x = \frac{1}{2}$ in $\ddot{x} = 3 - 4x$

$\therefore (\ddot{x})_{x=\frac{1}{2}} = 3 - 4 \times \frac{1}{2}$

$= 1 \text{ m/s}^2$

Point of Interest

The positive acceleration at $x = \frac{1}{2}$ indicates that the particle having come to rest at $x = \frac{1}{2}$, changes direction and moves in the positive direction

c). Let $f(x) = x^3 - 3x - 10$

The first approximation to the solution of $x^3 - 3x - 10 = 0$ is $x_1 = 2.7$

Using Newton's method the second approximation x_2 is given by

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$f(x) = x^3 - 3x - 10$

$f'(x) = 3x^2 - 3$

$f(2.7) = 2.7^3 - 3 \times 2.7 - 10$

$= 1.583$

$f'(2.7) = 3 \times 2.7^2 - 3$

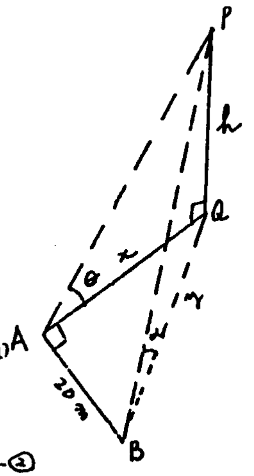
$= 18.87$

$\therefore x_2 = 2.7 - \frac{1.583}{18.87}$

$= 2.62$ (2 d.p).

Question No. 3

(a). Let $AQ = x$,
$BQ = y$
$PQ = h$
$\angle PAQ = \theta$
$\angle PBQ = \alpha$



$\cot P\hat{A}Q = \frac{3}{10}$

$\cot P\hat{B}Q = \frac{1}{2}$

From ΔAPQ; $\tan\theta = \frac{h}{x} = \frac{10}{3}$

$\therefore h = \frac{10}{3}x$ —— (1)

From ΔBPQ; $\tan\alpha = \frac{h}{y} = \frac{2}{1}$

$\therefore h = 2y$ —— (2)

From Eqⁿ (1) + (2), $\frac{10}{3}x = 2y$

$\therefore \boxed{y = \frac{5}{3}x}$ —— (3)

From ΔABQ, $x^2 + 20^2 = y^2$

ie $x^2 + 400 = \frac{25}{9}x^2$

ie $9x^2 + 3600 = 25x^2$

$\therefore 16x^2 = 3600$

$\therefore x^2 = 225$

$\therefore x = 15$ m

$\therefore h = \frac{10}{3}x$

$= \frac{10}{3} \times 15$ m

$= 50$ m

$\therefore$ The height of the tower is 50 m

(b). $f(x) = \frac{1}{2}\cos^{-1}(\sqrt{3}x)$

(i) $f(\frac{1}{2}) = \frac{1}{2}\cos^{-1}(\sqrt{3} \times \frac{1}{2})$

$= \frac{1}{2}\cos^{-1}\frac{\sqrt{3}}{2}$

$= \frac{1}{2} \times \frac{\pi}{6}$

$= \frac{\pi}{12}$

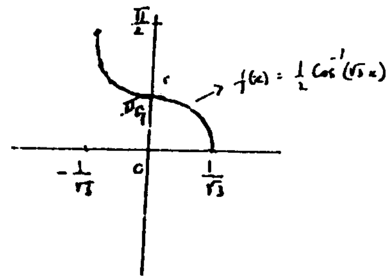
(ii). Domain:

D: $-1 \le \sqrt{3}x \le 1$

ie D: $-\frac{1}{\sqrt{3}} \le x \le \frac{1}{\sqrt{3}}$

(iii) The values of $f(x)$ for which the function is defined is. $0 \le f(x) \le \frac{\pi}{2}$ [or. Range: $\frac{0}{2} \le \frac{y}{2} \le \frac{\pi}{2}$ ie R: $0 \le f(x) \le \frac{\pi}{2}$]

(iv).



c). (i) Initial deposit = \$200

Rate of Interest = 9% p.a
= 0.75% p.m

Period = 3 yrs
= 36 months

Using $A = P(1 + \frac{r}{100})^n$

$\therefore$ Amount accrued $= 200\left(1 + \frac{0.75}{100}\right)^{36}$

$= 200 \times 1.0075^{36}$

(ii). No. of years = 12
∴ Period = 12×12
= 144 months

Rate = 0.75% p.m.

Amount invested every month = \$200

Amount accrued by the 1^{st} \$200 after 144 months = $\$200 \times 1.0075^{144}$
Amount accrued by the 2^{nd} \$200 " 143 months = $\$200 \times 1.0075^{143}$

. . . .

Amount accrued by the last \$200 . . . = $\$200 \times 1.0075^{1}$

∴ Amount in the fund after 12 years

$= \$\ 200\times1.0075^{1} + 200\times1.0075^{2} + \cdots + 200\times1.0075^{144}$

This is a G.P whose 1^{st} term is 200×1.0075 and whose common ratio is 1.0075

Using $S_n = \dfrac{a(r^n-1)}{r-1}$

$$S_{144} = \frac{200\times1.0075\left(1.0075^{144}-1\right)}{1.0075-1}$$

$$= \$\ 51928.88$$

∴ The amount in the fund after 12 years = \$ 51 928.88

NOTE: You may also solve the above problem using the established formulae. However, the above method is the preferred one.

Question no. 4

a) To prove by mathematical Induction

that $2 + 2^2 + 2^3 + \cdots + 2^n = 2(2^n - 1)$

NOTE: In this case No other form of proof is acceptable.

When $n=1$, LHS $= 2$
RHS $= 2$ } which is True [1]
∴ True for $n = 1$

Assume true for $n = k$

$2 + 2^2 + 2^3 + \cdots + 2^k = 2(2^k - 1)$ ——— ① [1]

Investigate whether true for $n = k+1$ [½]

$$\therefore\ 2 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} = 2(2^k - 1) + 2^{k+1}$$
$$= 2^{k+1} - 2 + 2^{k+1}$$
$$= 2 \cdot 2^{k+1} - 2$$
$$= 2(2^{k+1} - 1)$$ [1½]

This is of the same form as ①, when k is replaced by $(k+1)$.

∴ If the statement is true for $n = k$, then it is true for $n = k+1$

It is true for $n = 1$
∴ It must be true for $n = 2, 3$ and so on
∴ True for all n, $n \geq 1$, n integral.
∴ $2 + 2^2 + 2^3 + \cdots + 2^n = 2(2^n - 1)$ [1]

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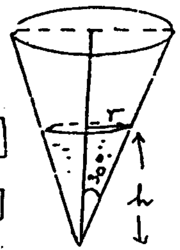
b) Radius of surface of water is given by $\frac{r}{h} = \tan 30°$
∴ $r = h \cdot \frac{1}{\sqrt{3}}$ [2]



∴ Volume of water $= \frac{1}{3}\pi r^2 h$ —[1]
$= \frac{1}{3} \times \pi \cdot \frac{h^2}{3} \cdot h$
$= \frac{\pi h^3}{9}$ metre³ [1]

$$\frac{dV}{dt} = 0.2\sqrt{h} \quad m^3/mt.$$ [1]

$$\frac{dh}{dt} = \frac{dh}{dV}\cdot\frac{dV}{dt} \quad \text{——— (1)}$$ [1]

Now, $V = \dfrac{\pi h^3}{9}$

$\therefore \dfrac{dV}{dh} = \dfrac{3\pi h^2}{9}$

$= \dfrac{\pi h^2}{3}$ [1]

$\therefore \dfrac{dh}{dt} = \dfrac{3}{\pi h^2} \times 0.2\sqrt{h}$

$= \dfrac{0.6\sqrt{h}}{\pi h^2}$ [1]

i.e. $\boxed{\dfrac{dh}{dt} = \dfrac{0.6}{\pi h^{3/2}}}$ m/mt.

(iv). When $h = 4$ m.

$\left(\dfrac{dh}{dt}\right)_{h=4m} = \dfrac{0.6}{\pi \times 4^{3/2}}$ [1]

$= \dfrac{0.6}{8\pi}$ m/mt

$\therefore \boxed{\dfrac{dh}{dt} = \dfrac{3}{40\pi} \text{ m/mt or } \dfrac{0.075}{\pi}}$ [1]

$\therefore$ Rate of reduction in depth when $h = 4$m $= \dfrac{3}{40\pi}$ m/mt.

(NB either a fraction or a decimal but a mixture of both not considered simplest form.)

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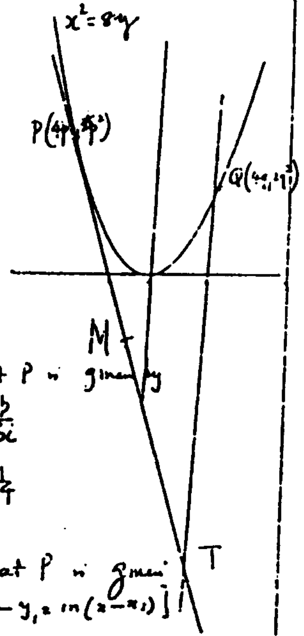
Question No. 5



At P:

$x = 4p$ }
$\therefore \dfrac{dx}{dp} = 4$,

$y = 2p^2$ }
$\therefore \dfrac{dy}{dp} = 4p$

$\therefore$ grad. of tangent at P is given by

$\dfrac{dy}{dx} = \dfrac{dy}{dp}\cdot\dfrac{dp}{dx}$

$= 4p \cdot \frac{1}{4}$

$= p$.

$\therefore$ Eq^n of tangent at P is given by [using $y - y_1 = m(x - x_1)$]

$y - 2p^2 = p(x - 4p)$

$\therefore y = 2p^2 + px - 4p^2$

i.e. $y = px - 2p^2$

(ii) Equation of QT is $x = 4q$ ——— (1)
Equation of PT is $y = px - 2p^2$ ——— (2)

$\therefore$ Solving (1) & (2), simultaneously:

$y = 4pq - 2p^2$

$\therefore$ Co-ords of T are: $(4q,\ 4pq - 2p^2)$

(iii) Co-ords of M are:

$\left(\dfrac{4p + 4q}{2},\ \dfrac{2p^2 + 4pq - 2p^2}{2}\right)$

$= [2(p+q),\ 2pq]$

(iv) Since M is the Mid-pt of PT, its co-ordinates are given by

$x = 2(p+q)$ ——— (1)

and $y = 2pq$ ——— (2)

Since $pq = -1$

$y = -2$

But, $y = -2$ is the equation of the directrix of the parabola $x^2 = 8y$

$\therefore$ The Cartesian Equation of the Locus of M is $y = -2$

b) $2\sin^2\theta + 3\cos\theta = 0$

$\therefore 2(1 - \cos^2\theta) + 3\cos\theta = 0$

$\therefore 2 - 2\cos^2\theta + 3\cos\theta = 0$

i.e. $2\cos^2\theta - 3\cos\theta - 2 = 0$

i.e. $(2\cos\theta + 1)(\cos\theta - 2) = 0$

$\therefore \cos\theta = -\frac{1}{2}, 2$

$\cos\theta = 2$ does not exist.

$\therefore \cos\theta = -\frac{1}{2}$

$\therefore \alpha = \frac{2\pi}{3}$

S	A
T	C

$\therefore$ The most general solution is $\theta = 2n\pi \pm \frac{2\pi}{3}$

$n = 0, 1, 2, \cdots$

Question No. 6

a) $\dfrac{dN}{dt} = k(N - 2000)$

$\therefore \dfrac{dN}{N-2000} = k\,dt$

$\therefore \displaystyle\int \frac{dN}{N-2000} = \int k\,dt + C$

$\therefore \log_e(N - 2000) = kt + C$ C = Const

$\therefore N - 2000 = e^{kt+C}$

$\therefore N - 2000 = Ae^{kt}$ A = const

(i) $\therefore N = 2000 + Ae^{kt}$ ——— (1)

Initially, number of sheep = 5000, i.e. when $t = 0$, $N = 5000$

$\therefore$ Sub. in (1).

$5000 = 2000 + Ae^{0}$

i.e. $5000 = 2000 + A$

$\therefore A = 3000$

$\therefore N = 2000 + 3000e^{kt}$

(ii) When $t = 2$ yrs, $N = 6000$

i.e. $6000 = 2000 + 3000e^{2k}$

$\therefore 3000e^{2k} = 4000$

$\therefore e^{2k} = \frac{4}{3}$

$\therefore 2k = \log_e \frac{4}{3}$

$\therefore k = \frac{1}{2}\log_e \frac{4}{3}$

$= 0.14384104$

(iii) When $t = 3$ yrs:

$N = 2000 + 3000e^{3 \times 0.14384104}$

$= 6618$

$\therefore$ Number of sheep after 3 years would be 6618 (iv). When N reaches 10,000,

$10{,}000 = 2000 + 3000\,e^{0.14384104t}$

$\therefore e^{0.14384104t} = \frac{8}{3}$

$\therefore 0.14384104t = \log_e \frac{8}{3}$

$\therefore t = \frac{1}{0.14384104}\log_e \frac{8}{3}$

$= 6.82$ yrs.

$\therefore$ In about 7 years, the sheep population would reach 10,000

(b) $x = 4\sin 2t + 3\cos 2t$

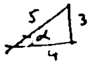
(i) $\therefore x = \sqrt{4^2+3^2}\left[\frac{4}{\sqrt{4^2+3^2}}\sin 2t + \frac{3}{\sqrt{4^2+3^2}}\cos 2t\right]$

$\therefore x = 5\left[\frac{4}{5}\sin 2t + \frac{3}{5}\cos 2t\right]$

This is of the form

$x = A\sin(2t+\alpha)$

When $A = 5$ and where either $\cos\alpha = \frac{4}{5}$ or $\sin\alpha = \frac{3}{5}$ or $\tan\alpha = \frac{3}{4}$



and where $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$.

(ii) $x = 5\sin(2t+\alpha)$.

$\dot{x} = 10\cos(2t+\alpha)$

$\ddot{x} = -20\sin(2t+\alpha)$

ie $\ddot{x} = -4\times 5\sin(2t+\alpha)$

ie $\ddot{x} = -4x$

This is of the form $\ddot{x} = -n^2x$

$\therefore$ The motion is Simple Harmonic

(iii) $n^2 = 4$

$\therefore n = 2$

$\therefore$ The Period of Oscillation is $T = \frac{2\pi}{n} = \frac{2\pi}{2}$

$= \pi$ seconds

(iv) Maximum displacement $= 5$

(v). To calculate velocity using $v^2 = n^2(a^2 - x^2)$

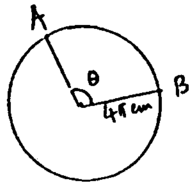
$v^2 = 4(5^2 - 3^2)$

$= 4\times 16$

$\therefore v = 8$ cm/s

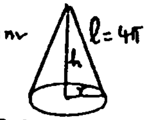
Question no 7

a).



Arc $AB = 4\pi\theta$

This arc will now form the base of the cone



The circumference of the base of the cone will have the same length as arc AB

$\therefore 2\pi r = 4\pi\theta$

$\therefore r = 2\theta$

Let the slope of the cone be l.

$\therefore l = 4\pi$ cm

Let the height of the cone be $= h$ cm

$\therefore h^2 = l^2 - r^2$

ie $h^2 = 16\pi^2 - 4\theta^2$

$\therefore h = \sqrt{4(4\pi^2 - \theta^2)}$

$= 2\sqrt{4\pi^2 - \theta^2}$

$\therefore$ Volume of Cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\cdot(2\theta)^2\cdot 2\sqrt{4\pi^2-\theta^2}$

$\therefore V = \frac{8\pi\theta^2}{3}\sqrt{4\pi^2-\theta^2}$

(ii) For Maximum Volume $\frac{dV}{d\theta} = 0$.

$V = \frac{8\pi\theta^2}{3}\sqrt{4\pi^2-\theta^2}$

$\frac{dV}{d\theta} = \frac{8\pi\theta^2}{3}\cdot\frac{1}{2}(4\pi^2-\theta^2)^{-1/2}\cdot(-)2\theta + (4\pi^2-\theta^2)^{1/2}\cdot\frac{16\pi\theta}{3}$

$= 0$ for max. volume

$\therefore \frac{8\pi\theta^3}{3}\times\frac{1}{\sqrt{4\pi^2-\theta^2}} = \frac{16\pi\theta}{3}\sqrt{4\pi^2-\theta^2}$

$\theta = 0$ is not a solution

$\therefore \frac{8\pi\theta^2}{3}\times\frac{1}{\sqrt{4\pi^2-\theta^2}} = \frac{16\pi}{3}\sqrt{4\pi^2-\theta^2}$

ie $\theta^2 = 2(4\pi^2-\theta^2)$

ie $\theta^2 = 8\pi^2 - 2\theta^2$

$\therefore 3\theta^2 = 8\pi^2$

$\therefore \theta^2 = \frac{8}{3}\pi^2$

$\therefore \theta = \sqrt{\frac{8}{3}}\,\pi$

$\therefore$ Max volume is obtained when $\theta = \sqrt{\frac{8}{3}}\cdot\pi$.

The Minimum value is zero.

Maximum Volume

$= \frac{8\pi}{3}\left(\sqrt{\frac{8}{3}}\cdot\pi\right)^2\cdot\sqrt{4\pi^2 - \left(\sqrt{\frac{8}{3}}\pi\right)^2}$

$= \frac{8\pi}{3}\times\frac{8\pi^2}{3}\cdot\sqrt{4\pi^2 - \frac{8\pi^2}{3}}$

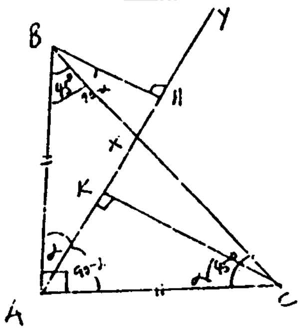
$= \frac{64\pi^3}{9}\sqrt{\frac{4\pi^2}{3}}$

$= \frac{128\pi^4}{9\sqrt{3}}$ cm³

$= \frac{128\sqrt{3}}{27}\pi^4$ cm³.

Note: You are not expected to work out the maximum volume. It is done here, only for interests' sake.

(vi)



DATA:- $\angle BAC = 90^\circ$
$AB = AC$
$\angle AKC = 90^\circ$
$\angle BHY = 90^\circ$

TO PROVE:- $AH = CK$

PROOF: Let $\angle BAH = \alpha^\circ$

Then: $\angle ABH = 90^\circ - \alpha^\circ$

$\angle CAK = 90^\circ - \alpha^\circ$

$\angle ACK = \alpha^\circ$

In Δ^s ABH, ACK,

$AB = AC$ —— data

$\angle BAH = \angle ACK$ —— $(= \alpha^\circ)$

$\angle ABH = \angle CAK$ —— $(= 90 - \alpha^\circ)$

$\therefore \Delta ABH \equiv \Delta ACK$ (AAS).

$\therefore AH = CK$

QED.

In Δs ABH and ACK

1. $AB = AC$ Given

2. $\angle BHA = \angle AKC = 90^\circ$ (BH & CK are given perpendicular to AY)

3. $\angle BAK + \angle KAC = 90^\circ$ ($\angle BAC = 90$ given)

$\therefore \angle KAC = 90 - \angle BAK = 90 - \angle BAH$ — ①

$\angle BAH + \angle AHB + \angle HBA = 180^\circ$ ($\angle$ sum Δ ABH)

$\therefore \angle HBA = 180 - (\angle BAH + \angle BHA)$
$= 180 - (\angle BAH + 90)$

$\angle HBA = 90 - \angle BAH$ — ②

From ① and ②
$\angle KAC = \angle HBA$

$\therefore \Delta ABH \equiv \Delta ACK$ (AAS)

$\therefore AH = CK$ (corresponding sides congruent Δs)